# Scale-free networks with a large- to hypersmall-world transition

Petter Holme [a]

[a] *Department of Computer Science, University of New Mexico, Albuquerque, NM 87131, U.S.A.*

**Abstract**

Recently there has been a tremendous interest in models of networks with a power-law distribution of degree—so called "scale-free networks." It has been observed that such networks, normally, have extremely short path-lengths, scaling logarithmically or slower with system size. As an exotic and counterintuitive example we propose a simple stochastic model capable of generating scale-free networks with linearly scaling distances. Furthermore, by tuning a parameter the model undergoes a phase transition to a regime with extremely short average distances, apparently slower than $\log \log N$ (which we call a hypersmall-world regime). We characterize the degree-degree correlation and clustering properties of this class of networks.

*Key words:* Complex Networks; Network Analysis; Network Dynamics; Scale-Free Networks
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## 1 Introduction

A major source of the recent surge of interest in complex networks has been the discovery that a large class of real-world networks have distributions of degree (the number of neighbors of a vertex) scaling like a power-law (1; 2; 3), so called *scale-free networks*. Ever since Ref. (4) there has been a tremendous number of works modeling networks with power-law degree distributions. One characteristic feature of most such model networks are that the distances (numbers of edges in shortest paths between vertices) are very short (so called *small-world networks* (3)), scaling like a logarithm, or like an even slower increasing function (5; 6), of the system size. It is however not true that all models with a power-law degree distribution have slowly increasing distances. In this paper we propose a simple, random network model having a power-law degree distribution with an arbitrary exponent, and a transition between

regimes of linearly scaling distances and distances scaling slower than a double logarithm. Our model is not as much a model of a real-world system as an example of vast variety of structure within the class of networks defined by a degree distribution.

## 2 The model

The models of scale-free networks can be divided into classes. Probably most proposed models are Markov chain growth (4; 7; 8; 9; 10; 11; 12) or equilibrium (13; 14; 15; 16) models where the power-law degree distribution is an emergent property of the system. In another class of models the degrees are treated as intrinsic properties of the vertices and thus preassigned before the edges are added. [1] Such models include the rather frequently used "configuration model" (19; 20), a model for networks with clear-cut core-periphery structure (21) and Internet at the largest scale (22). The model we propose belongs to the latter class.

Let $V$ be the set of $N$ vertices and $E$ be the set of $M$ edges. Our algorithm is defined as follows (details will be discussed below):

(1) Draw $n$ random integers (that will represent desired degrees) in the interval $[2, n^{1/(1-\gamma)}]$. Let the probability of picking $K$ be proportional to $K^{-\gamma}$.
(2) Sort the integers to an increasing sequence $K_1 \leq \cdots \leq K_n$.
(3) Go though the sequence in increasing index-order and for each vertex $i$:
   (a) With probability $1-p$, go through the vertices $[i+1, n]$ in increasing order and add an edge $(i, j)$ ($j \in [i+1, n]$) if the degrees of $i$ and $j$, $k_i$ and $k_j$, are lower than $K_i$ and $K_j$ respectively.
   (b) Otherwise (if step 3a is not realized, i.e. with probability $p$) go through the vertices $[i+1, n]$ in decreasing order and add an edge $(i, j)$ if $k_i < K_i$ and $k_j < K_j$.
(4) For every vertex with a degree $k_i$ less than its desired degree $K_i$, add $K_i - k_i$ one-degree vertices.

The total number of vertices $N$ will be $n$ plus the number of one-degree vertices added in step 4. By construction, the networks will have a power-law degree distribution for degrees of two or larger. The reason degree-one vertices are added in step 4 and not generated in step 1 is to give all high-degree vertices their desired degree. Without step 4 there would be many vertices with $K_i > k_i$

[1] There are also crossover models between these two classes—models where the degree distribution depend both on the a stochastic evolution process and intrinsic properties of the network (17; 5; 18).

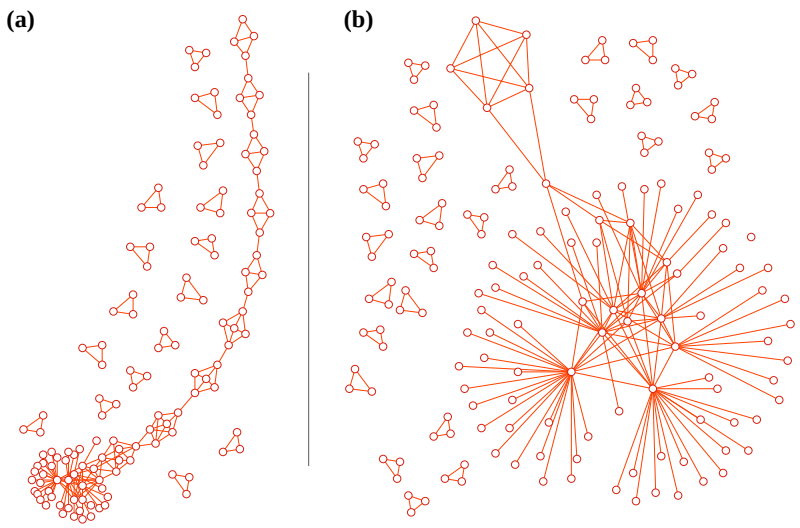


Fig. 1. Example networks with $n = 100$ and $p = 0$ (a) and $p = 0.01$ (b).

which would affect the high-degree vertices more than low-degree vertices. Since most power-law network models emphasize accurate modeling of the right end of the degree distribution, we add degree-one vertices last. Step. 2 is the computational bottleneck in the algorithm making the execution time of network construction $O(\max(n \log n, N))$ with a fast sorting algorithm.

## 3 Numerical results

In this section we will study the network structure numerically. We will use $10^4$ network realizations for the averages. For readability we use only one exponent of the degree distribution, $\gamma = 2.2$. The reason we choose this exponent is that it is we want a small exponent to make the network as broad (and unlike other sharper distributions) as possible; and, roughly speaking, 2.2 is the smallest exponent commonly seen in real world networks (for e.g. the Internet (2; 1)). We have checked all results for other exponents in the interval $2.2 < \gamma \leq 3$ too, and the conclusions will qualitatively hold for these values too. In general the networks can be composed of disconnected components (connected subgraphs). As common in such cases we will measure the quantities for the largest component. The scaling of the largest component size will also be discussed.

### *3.1 Example networks*

To get a feeling for the structure of the networks produced by the model two $n = 100$ networks are displayed in Fig. 1. The network in Fig. 1(a), with

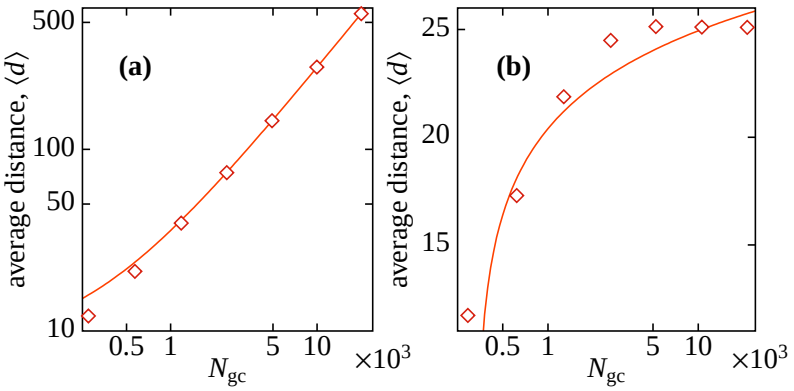


Fig. 2. The length scaling. (a) shows the average distance as a function of the size of the largest component for $p = 0$. The line is a fit to a linear form $8.2 \pm 1.2 + (0.0274 \pm 0.002)N$. (b) shows the corresponding curve for $r = 0.01$. The line is a fit to a general log log-form: $a_1 + a_2 \ln(a_3 + \ln(a_4 + N))$ where $a_1, \cdots, a_4$ are constants. Note the log-log scaled axes in (a) and log-lin scales in (b). Errorbars are smaller than the symbol size.

$p = 0$, is fragmented. The largest component has a chain-like shape. This can be understood from the construction algorithm. The major part of the degree-two vertices will form isolated triangles. Vertex $i$ attach to $i+1$ and $i+2$, then $i+1$ attaches to $i+2$ and all their desired degrees are reached. Starting from degree-three vertices a component will be formed. The degree-one vertices are connected to vertices of highest degree. If only a small fraction of the vertices are connected in reverse order the resulting network can be very different, see Fig. 1(b). Now almost the whole largest component is directly connected to a core of a dozen, or so, vertices. Like the $p = 0$ network there is a number of disconnected degree-two vertices.

### *3.2 Distance scaling*

Next we turn to the central quantity for our studies—the distance scaling. In Fig. 2(a) we plot the average distance as a function of the size of the largest component for $p = 0$. The scaling is, to a very high accuracy, linear. Other $\gamma$-values show the same qualitatively the scaling properties but the slope of the $d(N)$ curves is a function of $\gamma$ (lower $\gamma$ values have steeper slopes). This is in stark contrast to other scale-free network generators with length scaling like $\log N$, $\log N/\log\log N$ are even $\log\log N$ (6; 5). To interpret this we make two trivial observations about length scaling in network models with a fixed average degree: First, that the average distance in the network cannot increase faster than the maximal distance (the *diameter*). Second, that the diameter (and thus the average distance) cannot increase faster than linearly. One can thus say that the distances in our network model increase as fast as possible, given the average degree of the network. If $p$ is just a tiny bit larger than zero,

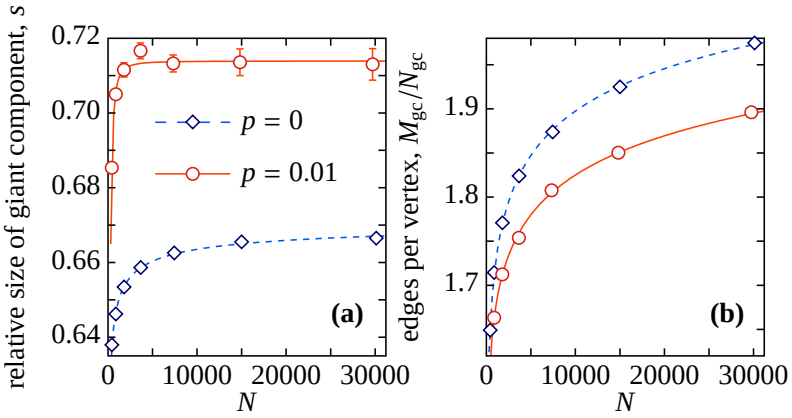


Fig. 3. (a) shows the relative sizes of the largest component $s$ as a function of system size. (b) shows the density of edges $M_{\rm gc}/N_{\rm gc}$ (half the average degree) in the largest component. Both curves are fits to algebraic decay forms.

the scenario is drastically different. The average distance scaling for $p = 0.01$ is plotted in Fig. 2(b). $d$ increases slower than a logarithmic function (a logarithmic growth would appear as a line in Fig. 2(b), due to the logarithmic $N$-axis). Indeed it seems to grow even slower than $\log\log N$. We see this by fitting the $d(N)$ values to a general $\log\log$-form $(a_1 + a_2 \ln(a_3 + \ln(a_4 + N))$ where $a_1, \cdots, a_4$ are constants)—the best fit is a function that increases significantly faster than the real curve. We also test bounded exponential and algebraic growth forms, but neither of these fits are extremely well to the observed curve. Indeed $d(N)$ appears to be bounded, or at least significantly slower increasing than $\log\log N$, which would mean our model have a transition from the theoretically maximal (linear) to the theoretically minimal (bounded) size scaling. With reference to the term "ultrasmall world" (6) we call sub-double-logarithmic scaling "hypersmall world." This conjecture would need further studies to be firmly established. Deriving the functional form is non-trivial—the probability that one reverse-adding (step 3b of the algorithm) should occur tends to one (as $1 - (1 - p)^n$) for any non-zero $p$. The first time this occurs, a star-graph of $O(n^{1/1-\gamma})$ vertices of high desired degree will be created. But the fraction of vertices in this star-graph goes to zero as $n^{-1/(1-1/\gamma)}$. After a few reverse-addings vertices in the middle of the spectrum of desired degrees will be saturated with edges and the range (in the ranking of vertices) between the high-degree vertex and the vertex being connected will grow rapidly. This increasing range is, phenomenologically, a possible cause for the extremely short distance-scaling.

### *3.3 Existence of a giant component*

As seen in Fig. 1 the model network can be disconnected. Although our model is intended as an example of the extreme diversity of network structure within

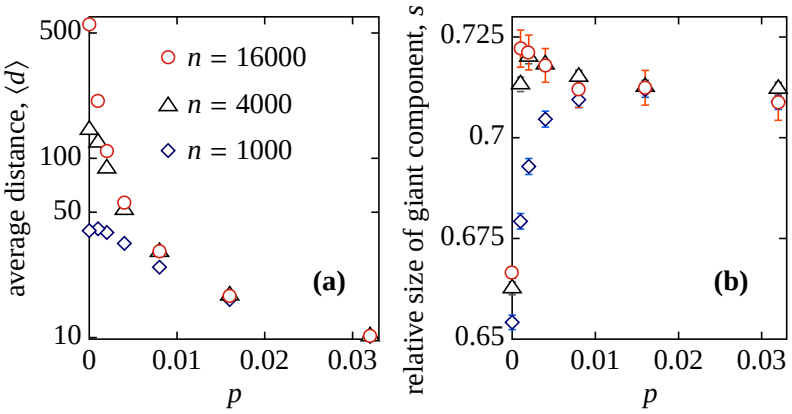


Fig. 4. The average distance in the largest component (a) and the average size of the giant component as functions of $p$.

the class of scale-free network, we would anyway like the largest component to grow at least linearly with the system size. In other words, there should be a giant component in the network. In Fig. 3(a) we plot the fraction of vertices in the largest component $s$. For both $p = 0$ and $p = 0.01$, $s$ seems to converge to a constant fraction of $N$. Note also that $s$ is (except one point in the $p = 0.01$ curve) an increasing function of $N$, and it is bounded by one. Other $p$ values show the same behavior. Thus we conclude that the model has, very likely, a giant component for all $p$ values. The density of edges in the giant component (which is half the average degree) plotted in Fig. 3(b). As the size of the giant component converges to $N_{\mathrm{gc}} = N_{\mathrm{gc}}^{\infty}$ the density of edges will be bounded by $M/N_{\mathrm{gc}}^{\infty}$. The convergence is rather slow, but consistent with an algebraic decay form.

### *3.4 Parameter dependence of distance and largest component size*

So far, we have established that our model has two fundamentally different network structures for $p = 0$ and $p = 0.01$. To complete this picture we study the distance scaling as a function of $p$. As seen in Fig. 3(a) the $\langle d \rangle(p)$-curves diverge very slowly for other than very small $p$-values. Nothing suggests that there would be more than two qualitatively different behaviors. Even though $p$ represent a kind of temperature-like disorder, it is hard to say if traditional methods of statistical physics (like renormalization group (23) or finite size scaling (24)) are valid in a model with the peculiar correlations induced by our construction scheme. To argue more speculatively, we observe that, initially, the curves' slopes decrease increasingly fast with $p$, but at an inflexion point the slope of a curve flattens out. This inflexion point $\tilde{p}$ seems to move toward $p = 0$ (for $n = 1000$ we have $0.004 \lesssim \tilde{p} \lesssim 0.008$, for $n = 4000$ we observe $0.002 \lesssim \tilde{p} \lesssim 0.004$, and for $n = 16000$ we $\tilde{p}$ seem to be less than 0.001), which is consistent with a $p = 0$ transition. There is always a chance $(1-p)^n$ that no

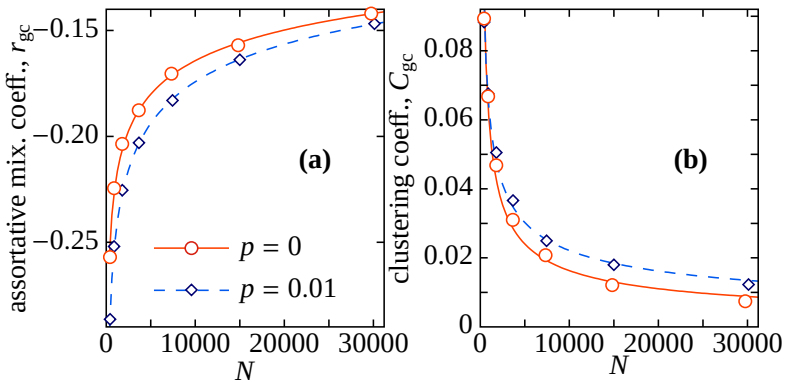


Fig. 5. The assortative mixing (a) and clustering (b) coefficients in the giant component. Standard errors are smaller than the symbol size. Both curves are fits to algebraic decay forms.

reverse-adding will occur during the construction. This is, we believe, the cause of the flattening of the $\langle d \rangle(p)$-curves as $p \searrow 0$. (For $n = 1$ and $p = 0.001$—the smallest non-zero $p$-value in Fig. 3(a))—absence of reverse-adding happen in $\sim 37\%$ of the network realizations.)

In Fig. 3(b) we plot the average size of the largest component, $s$, as a function of $p$. A strongly varying $s$ would make the distance scaling hard to interpret, but this is apparently not the case. Instead $s$ seems to converge as $p$ increases. The large-$p$ value is (for reasons do not speculate about in this paper) conspicuously size-independent.

### *3.5 Degree-degree correlations*

The degree distribution is perhaps the most fundamental network structure. A natural way to extend the characterization of the structure of a class of networks, from the degree distribution, is to ask how vertices of different degrees are interconnected. Is there a tendency for high degree vertices to attach to other high degree vertices, or do they preferably attach to low-degree vertices? A simple way to quantify this structure is to measure the linear correlation coefficient of degrees at either side of an edge, the *assortative mixing coefficient*: (3)

$$r = \frac{4\langle k_1\, k_2\rangle - \langle k_1 + k_2\rangle^2}{2\langle k_1^2 + k_2^2\rangle - \langle k_1 + k_2\rangle^2} \tag{1}$$

where $k_i$ is the degree of the $i$'th argument of the edges as they appear in an enumeration of the edges. This quantity takes values in the interval $[-1, 1]$, where low values mean that high-degree vertices primarily attach to low-degree vertices, and high values represent a tendency for vertices of high degree to

attach to one another. In both cases $r$ seems to converge algebraically to zero. This is seen by the accurate fits to algebraic decay forms, $b_1 N^{-b_2}$ ($b_1$ and $b_2$ are constants) in Fig. 5 (b). This convergence to $r = 0$ is more interesting when $p = 0$ because, by construction, except degree-one vertices, all vertices are attached to other vertices of quite similar values of degree. This means there is a strong assortative mixing in networks without degree one vertices. We preliminary confirm that $r$ converges to a positive value if step 4 is omitted. This means the negative contribution of attachment of degree-one vertices (at least almost) counterbalance the positive contribution from the ordered attachment in step 3a. Note, though, that random networks constrained only to a power-law degree distribution have a negative assortative mixing coefficient (25). In other words, in with such a null-model the degree-degree correlations are effectively positive for our model in the $p = 0$ case.

### *3.6 Clustering coefficient*

Another commonly studied network structure is clustering, or the density of triangles, in the network. In Fig. 5 (b) we present values of the clustering coefficient $C$—the number of unique triangles (fully connected subgraphs of three vertices) normalized to $[0, 1]$ by dividing by the number of connected triples of vertices (i.e. also including e.g. $\{i, i', i''\}$ where $(i, i'), (i', i'') \in E$ but $(i'', i) \notin E$) and a combinatorial factor three (for details see Ref. (3)). For both $p = 0$ and $p = 0.01$ the values of $C$ of the giant component seem to converge to zero from above. Once again this can be explained by the addition of degree-one vertices. As noted above, the highest connected vertices are primarily connected to degree-one vertices. Just as for the assortative mixing coefficient the degree-one vertices plays a major role in the decreasing nature of $C$. Without step 3a, the clustering coefficient is an increasing function, approaching rather large values. Now since the difference is only the lack of degree-one vertices, the triplets $(I, J, K)$ where $k_I = k_K = 1$ and $k_J$ is very high are the main negative contribution to $C$ (i.e., connected triples that are not triangles).

## 4 Summary and conclusions

In this paper we have proposed a generative network model with a power-law degree distribution. As a model parameter is tuned the model undergoes a transition from a situation where the average distance of the giant component scale linearly with the system size $N$. Thus our model has a large-world regime (with super-logarithmic distance scaling). In contrast, almost all previous complex network models we are aware of belong to the opposite category—

small-world networks—with exponential, or sub-exponential, distance scaling. The network models we are aware of (26; 27; 28), that do have a large-world regime, are with one exception (8; 29) trivial regular lattices or circulants. We proceed to evaluate the structure of these model in both the large- and small-world region of the parameter space. The networks are disconnected even in the $N \to \infty$ limit, but they do have a giant component.

This model is primarily intended as an exotic example, illustrating the great variety of networks having a power-law degree distribution. Even though the ensemble of all networks with a power-law degree distribution never show distances growing faster than a logarithm, this does not apply to all network models with an emergent power-law degree distribution. Our conclusion is thus (like in e.g. Ref. (30)) that one cannot treat all network models generating power-law degree distribution as one class. Our model include randomness in different steps of the construction, but the sorting of vertices and sequential addition induce strong correlations that give rise to the exotic length scaling properties. The construction mechanism is not limited to power-law degree distributions. We believe the high-degree vertices of power-law networks are needed to create the hypersmall-world scaling (extremely short pathlengths need an almost fully connected core that is directly connected to most of the rest of the network—with too limited degrees, such a core would be far from fully connected). One may wonder why real-world networks with a power-law like degree distribution (at least all we know of) are not large-world networks. This is of course a question for every system individually, but part of the answer can be found in the transition found in the model. Only a tiny perturbance is needed to turn the large-world distance scaling into a small-world—the large-world regime is exotic, but it is also instable to small perturbances.

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